Exercise 3.4.2. Determine the matrix of change of basis, when the old basis is the standard basis $(e_1, e_2, \ldots, e_n)$ and the new basis $(e_n, e_{n-1}, \ldots, e_1)$.

Solution. $P = (e_n, \ldots, e_1)$.

Exercise 3.4.3. Determine the matrix P of change of basis when the old basis is (e_1, e_2) and the new basis is $(e_1 + e_2, e_1 - e_2)$.

Solution. Since $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we know $P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Exercise 3.4.9. Let $V = F^n$. Establish a bijective correspondence between the set $\mathscr{B}$ of basis of V and $GL_n(F)$.

Proof. For every ordered basis $(v_1, v_2, \ldots, v_n)$, we can map it to $\begin{bmatrix} v_1, \ldots, v_n \end{bmatrix} \in GL_n(F)$. □

Exercise 4.3.1. Let V be the vector space of real 2×2 symmetric matrices $X = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$, and let $A = \begin{bmatrix} 2 & 1 \\ & 1 \end{bmatrix}$. Determine the matrix of the linear operator on V defined by $X \mapsto AXA^t$, with respect to a suitable basis.

Solution. Note that $AXA^t = \begin{bmatrix} 4x + 4y + z & 2y + z \\ 2y + z & z \end{bmatrix}$. We can choose $\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$ as a basis, mapping each of them to $\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, respectively.