

Exercise 2.10.1. Let G be the group of invertible real upper triangular 2×2 matrices. Determine whether or not the following conditions describe normal subgroups H of G . If they do, use the First Isomorphism Theorem to identify the quotient group G/H .

(a) $a_{11} = 1$

(b) $a_{12} = 0$

(c) $a_{11} = a_{22}$

(d) $a_{11} = a_{22} = 1$

Solution. (a) YES. Consider $f : G \rightarrow \mathbb{R}^\times$, which maps A to a_{11} . Since $(AB)_{11} = a_{11} \cdot b_{11}$, f is a homomorphism and surjective with kernel H .

(b) NO. H is not normal. Select $A = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \in H$ and $B = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in G$, we have

$$BAB^{-1} = \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix} \notin H.$$

(c) YES. Consider $f : G \rightarrow \mathbb{R}^\times$, $A \mapsto \frac{a_{11}}{a_{22}}$. Then $\ker(f) = H$.

(d) YES. Consider $f : G \rightarrow G'$, $A \mapsto \begin{pmatrix} a_{11} & \\ & a_{22} \end{pmatrix}$, where G' is the group of 2×2 invertible diagonal matrices.

Exercise 2.10.3. Let P be a partition of a group G with the property that for any pair of elements A, B of the partition, the product set AB is contained entirely within another element C of the partition. Let N be the element of P which contains 1. Prove that N is a normal subgroup of G and that P is the set of its cosets.

Proof. Consider $\varphi : G \rightarrow P$, which maps every $a \in A$ to A . For any $a \in A, b \in B$, we have

$$\varphi(a)\varphi(b) = AB = C,$$

and

$$\varphi(ab) = C, \text{ since } ab \in C.$$

Note that N is the kernel of φ , so N is normal and $P = G/N$.

□

Exercise 2.10.5. Identify the quotient group $\mathbb{R}^\times/P$, where P denotes the subgroup of positive real numbers.

Solution. Consider the function $f : \mathbb{R}^\times \rightarrow \langle \{\pm 1\}, \times \rangle$ which is defined as follows:

$$f(x) = \begin{cases} 1, & x > 0; \\ -1, & \text{otherwise.} \end{cases}$$

It is clear that f is a homomorphism with kernel P , thus $\mathbb{R}^\times/P \cong \langle \{\pm 1\}, \times \rangle$.

Exercise 2.10.6. Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth roots of unity. Describe the cosets of H in G explicitly, and prove that G/H is isomorphic to G .

Proof. $H = \{e^{i\frac{\pi}{2}k} : k = 0, 1, 2, 3\}$. For any $a \in G$, a can be represented as $e^{i\alpha+\beta}$, $\alpha \in [0, 2\pi)$, $\beta \in \mathbb{R}$, we have $aH = \{e^{i(\frac{\pi}{2}k+\alpha)+\beta} : k = 0, 1, 2, 3\}$.

Consider $f : G \rightarrow \mathbb{C}^\times$, $f(x) = x^4$, then f is a surjection and a homomorphism, which implies $\mathbb{C}^\times$ is isomorphic to G/H .

□